

Combined local implementation of nonlocal operations using GHZ states

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We propose a protocol for local implementation of two consecutive nonlocal operations by three parties. It consumes one shared GHZ state in this protocol. We also demonstrate that these resources are sufficient and necessary to locally implement two consecutive CNOT operations.

PACS numbers: 03.67.Lx

I. INTRODUCTION

In distributed quantum computation, collective operations often need to be implemented on the qubits at distant nodes. Such operations can only be implemented locally, i.e., it can only be implemented using local operations and classical communications (LOCC), shared entanglement and some auxiliary qubits. If the circuit of the operation has been constructed in one node, a straightforward method to implement such a nonlocal operation is using quantum state teleportations [1], i.e., teleporting all of the qubits to this node, performing the operation and teleporting these qubits back.

In the bipartite case, two rounds of state teleportation are required using this method. So in this case, it consumes two ebits (shared entanglement resources). However, there are operations that can be locally implemented using less entanglement resources [2, 3]. It has been demonstrated theoretically [2] and experimentally [4] that in order to implement a CNOT operation, one ebit is necessary and sufficient. In fact, any diagonal or offdiagonal block operation can be implemented using the same resources, even if the detail of the blocks are unknown [11]. Recently, the problems of constructing a nonlocal operation or simulating it by other operations are discussed [5, 6, 7, 8, 9]. In this paper, we only consider the case that the circuit of the operation has been locally constructed in one node. Of course, if the operation is completely known, it can also be implemented using the simulating methods discussed in the above references. But, the parties must construct new circuits to use these methods. These accessory devices need consume more local resources and may bring a loss of accuracy.

In the multiparty case, effective protocols for some operations are also proposed recently [2, 11]. The entanglement resources used in these protocols are Bell states. Nevertheless, in multiparty case, the GHZ states are also important and useful. So it is also interesting to search effective protocols using GHZ states.

There are some similarities between the local implementation of nonlocal operations and the remote implementation of local operations. Usually, a protocol for remote implementation of operations with some forms can also be used to locally implement nonlocal block operations with similar forms [11]. Ref. [10] proposed a protocol for consecutively remote implementation of two diagonal operations from two distant nodes to one node, using a shared GHZ state. This method can also be used in local implementation of nonlocal operations. We will discuss this issue in the following sections.

In Sec. II, we propose a protocol to locally implement two consecutive diagonal block operations among three parties — the first is on Alice's qubit A and Bob's qubit B , the second is on A and Charlie's qubit C . It requires one shared GHZ state, and it is available even if the detail of the blocks are unknown. The CNOT operation is a diagonal block operation, so two consecutive CNOTs can also be implemented using the protocol. We prove that these resources (one shared GHZ state) are also necessary for consecutive two CNOT operations. In Sec. III, we summarize our results.

II. THE PROTOCOL

Three distant parties Alice, Bob and Charlie have three qubits A , B and C respectively. Alice and Bob need to implement a CNOT on qubits A and B . Following the protocol in Ref. [2], Alice and Bob can accomplish this task using one shared Bell state (one ebit) — the necessary and sufficient entanglement resources for this task. Then consecutively, Alice and Charlie need to implement a CNOT on qubits A and C . Similarly, Alice and Charlie

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need another shared Bell state to accomplish this task. Thus, if considering these two operations as a whole, using above method, the three parties need a total of two shared Bell state — one shared by Alice and Bob, the other shared by Alice and Charlie. However, they can also implement these two CNOT operations using only one GHZ state shared by the three parties. We will propose such a protocol in this section. This protocol can be used to consecutively implement any two diagonal block operations even if the detail of the blocks are unknown. Since the CNOT operation is also a diagonal block operation, we will discuss the general case firstly. Then we will discuss the necessary entanglement resources for two consecutive CNOTs.

Alice, Bob and Charlie need to implement two nonlocal operations in turn. Firstly, Alice and Bob need to implement a two-qubit diagonal block operation U on their qubits A and B , where

$$U = \begin{pmatrix} u_0 & 0 \\ 0 & u_1 \end{pmatrix} = \sum_{i=0}^1 |i\rangle\langle i| \otimes u_i, \quad (1)$$

where u_0 and u_1 are 2×2 unitary matrices. Bob has the device of the operation U , but he may only know the form of it (Eq. 1), and may not know the detail of u_0 and u_1 . Then, Alice and Charlie need to implement a two-qubit diagonal block operation V on their qubits A and C , where

$$V = \begin{pmatrix} v_0 & 0 \\ 0 & v_1 \end{pmatrix} = \sum_{i=0}^1 |i\rangle\langle i| \otimes v_i, \quad (2)$$

where v_0 and v_1 are 2×2 unitary matrices. Charlie has the device of the operation V , but he may only know the form of it (Eq. 2), and may not know the detail of v_0 and v_1 . Thus, as a result, they need to implement a three-qubit operation

$$\begin{aligned} W &= \left(\sum_{i=0}^1 |i\rangle\langle i| \otimes I \otimes v_i \right) \left(\sum_{j=0}^1 |j\rangle\langle j| \otimes u_j \otimes I \right) \\ &= \sum_{i=0}^1 |i\rangle\langle i| \otimes u_i \otimes v_i \end{aligned} \quad (3)$$

on their qubits A, B, C .

These two nonlocal operation can be locally implemented in turn using the protocol proposed in Ref. [11]. Using this method, they need two Bell states — one shared by Alice and Bob, the other shared by Alice and Charlie. We will prove in the followings that they can also locally implement these two operations using one GHZ state shared by the three parties.

Initial state The initial state of the qubits A, B, C can always be expressed as

$$|\Psi_0\rangle_{ABC} = \alpha_0 |0\rangle_A |\xi_0\rangle_{BC} + \alpha_1 |1\rangle_A |\xi_1\rangle_{BC}, \quad (4)$$

where $|\xi_0\rangle$ and $|\xi_1\rangle$ can be any two-qubit state and are not orthogonal in general.

They share a GHZ state in qubits $A_1 B_1 C_1$

$$|\Phi\rangle_{A_1 B_1 C_1} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (5)$$

where qubit A_1 belongs to Alice, qubit B_1 belongs to Bob and qubit C_1 belongs to Charlie.

They can use the protocol specified as the following steps to accomplish their tasks.

step 1 Alice performs a CNOT operation on her qubits A and A_1 , using the qubit A as the control. After this, the state of A, B, C, A_1, B_1, C_1 becomes

$$\begin{aligned} & CNOT^{A, A_1} |\Psi_0\rangle_{ABC} \otimes |\Phi\rangle_{A_1 B_1 C_1} \\ &= CNOT^{A, A_1} \sum_i \alpha_i |i\rangle_A |\xi_i\rangle_{BC} \frac{1}{\sqrt{2}} \sum_j |jjj\rangle_{A_1 B_1 C_1} \\ &= \frac{1}{\sqrt{2}} \sum_{ij} \alpha_i |i\rangle_A |\xi_i\rangle_{BC} |i \oplus j\rangle_{A_1} |jj\rangle_{B_1 C_1}, \end{aligned} \quad (6)$$

where “ \oplus ” denotes the addition module 2.

Then she measures A_1 in computational basis $|a\rangle\langle a|$, ($a = 0, 1$), and tell the result a to Bob and Charlie. The state of A, B, C, B_1, C_1 becomes

$$\begin{aligned} & \sum_{ij} \alpha_i |i\rangle_A |\xi_i\rangle_{BC} |jj\rangle_{B_1 C_1} \delta_{a, i \oplus j} \\ &= \sum_{ij} \alpha_i |i\rangle_A |\xi_i\rangle_{BC} |jj\rangle_{B_1 C_1} \delta_{j, i \oplus a} \\ &= \sum_i \alpha_i |i\rangle_A |\xi_i\rangle_B |i \oplus a\rangle_{B_1} |i \oplus a\rangle_{C_1}. \end{aligned} \quad (7)$$

step 2 If the result $a = 0$ Bob does nothing, if $a = 1$ Bob performs the operation X on B_1 , where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (8)$$

is the first Pauli matrix. Because $X|i\rangle = |i \oplus 1\rangle$, ($i = 0, 1$), the state of A, B, C, B_1, C_1 becomes

$$\sum_i \alpha_i |i\rangle_A |\xi_i\rangle_{BC} |i\rangle_{B_1} |i \oplus a\rangle_{C_1}. \quad (9)$$

step 3 Bob performs the two-qubit operation U on his qubits B_1 and B . The state of A, B, C, B_1, C_1 becomes

$$\begin{aligned} & U^{B_1, B} \sum_i \alpha_i |i\rangle_A |\xi_i\rangle_{BC} |i\rangle_{B_1} |i \oplus a\rangle_{C_1} \\ &= \sum_i \alpha_i |i\rangle_A [(u_i \otimes I) |\xi_i\rangle]_{BC} |i\rangle_{B_1} |i \oplus a\rangle_{C_1}. \end{aligned} \quad (10)$$

step 4 Bob performs the operation H on B_1 , where

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (11)$$

is the Hadamard operation. Because $H|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^i |1\rangle)$, the state of A, B, C, B_1, C_1 becomes

$$\begin{aligned} & \sum_i \alpha_i |i\rangle_A [(u_i \otimes I) |\xi_i\rangle]_{BC} \frac{1}{\sqrt{2}} (|0\rangle + (-1)^i |1\rangle)_{B_1} |i \oplus a\rangle_{C_1} \\ &= \frac{1}{\sqrt{2}} |0\rangle_{B_1} \sum_i \alpha_i |i\rangle_A [(u_i \otimes I) |\xi_i\rangle]_{BC} |i \oplus a\rangle_{C_1} \\ & \quad + \frac{1}{\sqrt{2}} |1\rangle_{B_1} \sum_i (-1)^i \alpha_i |i\rangle_A [(u_i \otimes I) |\xi_i\rangle]_{BC} |i \oplus a\rangle_{C_1}. \end{aligned} \quad (12)$$

Bob measures B_1 in the computational basis $|b\rangle\langle b|$, ($b = 0, 1$), and tell the result b to Alice. The state of A, B, C, C_1 becomes

$$\sum_i (-1)^{ib} \alpha_i |i\rangle_A [(u_i \otimes I) |\xi_i\rangle]_{BC} |i \oplus a\rangle_{C_1}. \quad (13)$$

step 2' Charlie does the same thing on C_1 as Bob does on B_1 in step 2.

step 3' Charlie performs the two-qubit operation V on his qubits C_1 and C .

step 4' Charlie does the same thing on C_1 as Bob does on B_1 in step 4. Then he tells Alice the measurement result c . Obviously, Charlie's operations in step 2'-4' and Bob's operations in step 2-4 are not correlative. So, they can do these operations in parallel.

Similarly, after step 2'-4', the state of A, B, C becomes

$$\sum_i (-1)^{i(b+c)} \alpha_i |i\rangle_A [(u_i \otimes v_i) |\xi_i\rangle]_{BC}. \quad (14)$$

step 5 If $b = c$ ($b \oplus c = 0$) Alice does nothing, if $b \neq c$ ($b \oplus c = 1$) Alice performs the operation Z on A , where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (15)$$

is the third Pauli matrix. Because $Z|i\rangle = (-1)^i|i\rangle$, the state of A, B becomes

$$\sum_i \alpha_i |i\rangle_A [(u_i \otimes v_i) |\xi_i\rangle]_{BC}. \quad (16)$$

It can be directly calculated that

$$W|\Psi_0\rangle_{ABC} = \sum_i \alpha_i |i\rangle_A [(u_i \otimes v_i) |\xi_i\rangle]_{BC}. \quad (17)$$

Thus, after these steps, these two diagonal block operation U^{AB} and V^{AC} are determinately implemented on A, B, C using one shared GHZ state plus 3 cbits — one from Alice to Bob and Charlie, one from Bob to Alice and one from Charlie to Alice.

This protocol can be expressed as Fig. 1.

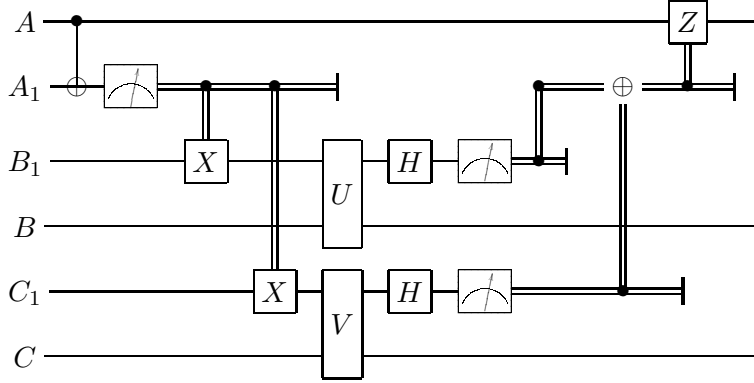


FIG. 1: Quantum circuit of the protocol using GHZ state, $A_1B_1C_1$ are in GHZ state defined by Eq. 5.

It can be found from the proof of the protocol that the protocol is independent on the dimension of the blocks. So this protocol is also available if the blocks have higher dimensions, i.e., u_0 and u_1 are $2^N \times 2^N$ unitary matrices, v_0 and v_1 are $2^M \times 2^M$ unitary matrices, and Bob and Charlie have the corresponding N and M qubits.

Obviously, the CNOT operation is a diagonal block operation. So two consecutive CNOT operations can be implemented using this method. In fact, it equals that the three parters implement a three-qubit operation

$$G = |0\rangle\langle 0| \otimes I \otimes I + |1\rangle\langle 1| \otimes X \otimes X \quad (18)$$

on their qubits A, B, C .

Suppose the three parters can implement G on ABC via some procedure using LOCC and shared entanglement resources. Because

$$G(|+\rangle_A |0\rangle_B |0\rangle_C) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC}, \quad (|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)), \quad (19)$$

so if the initial state of qubits ABC is $|+\rangle_A |0\rangle_B |0\rangle_C$, the three parters can finally get a GHZ state in qubits ABC after this procedure. Thus, they require to share entanglement resources more than one GHZ state in the beginning, as the amount of entanglement can not be increased by LOCCs.

Thus, our protocol is optimal for two consecutive CNOT operations. But, if they implement two CNOT consecutively using the protocol in Ref. [2], they require two shared Bell states. Obviously, they can get a shared GHZ state using these two Bell states via LOCC. If regarding Bob and Charlie as a whole, the amount of bipartite entanglement between Alice and them is one ebit in one GHZ state, and is two ebits in two Bell states. So, in this sense they need more entanglement resources than one shared GHZ state. Thus it is not optimal to use this method.

III. CONCLUSIONS

In this paper, we have addressed the problem of the local implementation of nonlocal operations by three parties — Alice, Bob and Charlie. We have proposed a protocol for local implementation of two consecutive two-qubit diagonal block operations — the first by Alice and Bob, the second by Alice and Charlie. It consumes one shared GHZ state in this protocol. However, they need two shared Bell state if they independently implement these two operations via the bipartite protocol in turn. It is interesting that such overall treatment can save resources. We have also demonstrated that one shared GHZ state is sufficient and necessary to locally implement two consecutive CNOT.

Acknowledgments

We acknowledge all the collaborators of our quantum theory group at the Institute for Theoretical Physics of our university. This work was funded by the National Natural Science Foundation of China under Grant No. 60573008.

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